The Barnett approximation, on the other hand, yields

$$p/p_0 = \tau^{-10/3 - 4/3 F^2} \exp\left({}^8/_9 F\tau - {}^2/_3 F\tau^{-1} - {}^4/_{27} F^2\tau^2\right)$$

and the solution is asymptotically stable for any fixed value of F.

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SOME CONTACT PROBLEMS FOR A THREE-DIMENSIONAL WEDGE WITH A FINITE NUMBER OF CONTACT REGIONS

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Mixed problems for a three-dimensional wedge whose edge is unbounded on both sides are considered. The case of several contact sections between the wedge and the stamps is investigated. Theorems for solvability of the integral equations are established in a number of cases and the properties of their solutions are studied. Approximate formulas are obtained for small wedge angles.

The problem was examined in [1] in the case of one contact section, where the method elucidated in [2] was applied. The convolution integral equation given on a system of segments was studied in [3].

The equation of [1] on a system of segments is considered below according to the scheme elucidated in [3].

1. On the basis of the Ufliand solution [4] the antisymmetric mixed problem (Problem 1) for an elastic three-dimensional wedge with $2N$ lines of separation of the boundary conditions, parallel to the wedge edge is investigated. The wedge edge agrees with the direction of the axis of the spatial coordinate z. Each pair of lines with the numbers $2n-1$ and $2n$ is separated a distance a_{2n-1} and a_{2n} $(n = 1, 2, \ldots, N)$ from the edge, respectively. The normal pressure $p\,(r, \pm\alpha, z)$ and the displacements parallel to the wedge faces, equal to zero, are assumed given in each region formed by such a pair of lines. The single unknown in the domain of contact is the normal displacement. The wedge faces outside the mentioned regions are rigidly clamped.

This problem, artificial to some degree, generates a new class of systems of integral equations which have not been investigated earlier. The mechanically natural problem about the aperiodic shear of a three-dimensional wedge by a system of stamps reduces to these same systems.

The case of harmonic vibrations (Problem 2) caused by this same system of stamps is among the second group of problems in connection with the qualitative distinction of the kernels of the corresponding integral equations.

It is assumed in problems about the aperiodic shear and the harmonic oscillations that one of the wedge faces is rigidly clamped, and the other is loaded by a system of stamps moving parallel to the spatial axis (coincident with the wedge edge, as in Problem 1). The stamp displacements are decribed in the harmonic and aperiodic cases by the functions

$$\operatorname{Re} f_n(r^*)\, e^{-i\omega t}, \qquad f_n(r^*)\, e^{-\varepsilon t}, \qquad a_{2n-1} \leqslant r^* \leqslant a_{2n} \quad (\varepsilon > 0,\ \omega > 0)$$

respectively. All the assumptions of Problem 1 are retained relative to the lines of interchange in the boundary conditions.

In the harmonic case the wedge material is considered viscoelastic with a time-constant Young's modulus E, Poisson's ratio ν and creep $\theta\,(t-\tau)$ dependent on the difference between the arguments of the kernel. The wedge is assumed elastic in the aperiodic case. Under the conditions mentioned, the contact stresses under the stamps are to be determined.

The problems described above are reduced to the solution of the following kind of integral equation:

$$Kq = \sum_{n=1}^{N} \int_{\alpha_{2n-1}}^{\alpha_{2n}} k\,(r, \rho)\; q\,(\rho)\, d\rho = \Delta f_m\,(r) \tag{1.1}$$

$$k\,(r, \rho) = \frac{1}{\pi i} \int_{-\infty}^{\infty} I_{-iu}\,(\varkappa r)\, K_{-iu}\,(\varkappa\rho)\, K\,(u)\, u\, du$$

$$\alpha_{2m-1} \leqslant r \leqslant \alpha_{2m}, \qquad m = 1, 2, \ldots, N$$

by using the Kantorovich-Lebedev transformation [4]. Here $I_\lambda\,(\varkappa r)$, $K_\lambda\,(\varkappa r)$ are modified Bessel functions.

The following notation is used in the case of Problem 1:

$$\Delta^{-1} = 4G\,(1-\nu), \qquad \alpha_k = \frac{a_k}{a}, \quad a = \sum_{n=1}^{N} (a_{2n} - a_{2n-1})$$
$$a_{2n} > a_{2n-1}, \qquad k = 1, 2, \ldots, 2N \tag{1.2}$$
$$f_m\,(r) = C_1\,(m)\; I_\mu\,(\varkappa r) + C_2\,(m) K_\mu\,(\varkappa r) + \varphi\,(r)$$

$$K\,(u) = u\,(\operatorname{ch} 2\alpha u - \cos 2\alpha)\,[(3-4\nu)\operatorname{sh} 2\alpha u + u \sin 2\alpha]^{-1}\,(u^2 + \mu^2)^{-1}$$

Here $C_1(m)$, $C_2(m)$ are constants to be determined, $\mu > 0$ is arbitrary, $\varphi(r)$ is a particular solution of the equation

$$B(\mu)\varphi = r\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) - (\varkappa^2 r^2 + \mu^2)\varphi = \Phi(r)$$

$q(r), \Phi(r)$ are the Fourier tranform in the coordinate z of the normal displacements and stresses, respectively, $\varkappa = |\zeta| > 0$, ζ is the Fourier transform parameter in the coordinate z and 2α is the wedge apex angle. The relation between the dimensionless parameters r, ρ and the dimensional parameters r^*, ρ^* is

$$r = r^* / a, \qquad \rho = \rho^* / a$$

It should be noted that the expression (1.2) has been obtained from

$$B(\mu)\sum_{n=1}^{N}\int_{\alpha_{2n-1}}^{\alpha_{2n}} k(r, \rho) q(\rho) d\rho = A\Phi(r) \tag{1.3}$$

which is the initial integral equation of the Problem 1. Equation (1.1) with a right side of the form (1.2) is obtained as a result of applying the operator $B^{-1}(\mu)$ to both sides of (1.3). The method described has been elucidated earlier in [2] and then has been applied in [1].

The following notation is used in the case of Problem 2:

$$\varkappa^2 = -D\sigma^2 a^2 G^{-1}, \quad K(u) = u^{-1}\,\mathrm{th}\,\alpha u, \quad A = G / a$$

$$\sigma^2 = \omega^2\theta^*, \qquad \theta^* = 1 + \int_0^\infty \theta(\tau) e^{i\omega\tau} d\tau, \qquad G^\circ = G\theta^*$$

where α is the wedge angle, D is the material density, $f(r)$ is the amplitude of the stamp displacement in the domain $\alpha_{2m-1} \leqslant r \leqslant \alpha_{2m}$, G° and ν are, respectively, the shear modulus and Poisson's ratio. The connection between the dimensionless and dimensional parameters is the same as in the static problem.

In the aperiodic case $i\omega$ should be replaced by ε in the last formulas and it should be assumed that $\theta(\tau) \equiv 0$.

The properties of the functions $K(u)$ and $K_{\pm}(u)$ are described in [1].

2. Applying the asymptotic estimates of the behavior of the functions $I_\lambda(\varkappa r), K_\lambda(\varkappa r)$ in the complex λ plane [1], we obtain the following estimate in the case $\varkappa > 0$:

$$k(r, \rho) = c \ln t\,[1 + o(1)], \quad t = |\ln(r / \rho)| \to 0 \tag{2.1}$$

Let us introduce the spaces $H_{0.5}(\Omega)$, $L_p(\Omega)$, $T(\Omega)$, $C(0.5)$ with the respective metrics

$$\|q\|_{H_{0.5}(\Omega)} = \left(\int_0^\infty |Q(u)|^2 K(u)\, u \,\mathrm{sh}\,\pi u\, du\right)^{1/2}$$

$$Q(u) = \int_\Omega q(r) K_{iu}(\varkappa r)\, dr, \qquad \Omega = \bigcup_{n=1}^{N}(\alpha_{2n-1}, \alpha_{2n})$$

$$\|q\|_{L_p(\Omega)} = \left(\int_\Omega |q(r)|^p dr\right)^{1/p}, \qquad p > 1$$

$$\|q\|_{C(0,5)} = \sup_n \sup_r |q(r)(r - \alpha_{2n-1})^{1/2}(\alpha_{2n} - r)^{1/2}|$$

$$\|q\|_{T(\Omega)} = \sup_n \sup_r |q(r)|, \quad r \in \Omega, \quad n = 1, 2, \ldots, N$$

The following theorems are valid.

Theorem 1. The operator K acts from $L_p(\Omega)$, $p > 1$ continuously into $T(\Omega)$.

The proof is based on (2.1).

Theorem 2. Any $L_p(\Omega)$, $p > 1$ is imbedded in the space $H_{0.5}(\Omega)$

The proof is based on using the integral representation of the function $K_{iu}(\varkappa r)$ [5] and the generalized Hausdorff-Young inequality [6].

Theorem 3. In the $\varkappa > 0$ case (1.1) has not more than one solution in the space $L_p(\Omega)$, $1 < p \leqslant 2$.

Indeed, for $q(r) \in L_p(\Omega)$, $1 < p \leqslant 2$ the relationship

$$\| q \|^2_{H_{0.5}(\Omega)} = \operatorname{Re} \int_\Omega f(r) \overline{q(r)}\, dr$$

is correct because of the estimate (2.1) and Theorems 1 and 2. Then, $q(r) \equiv 0$ results from the condition $f(r) \equiv 0$ for all $r \in \Omega$. This last result means that if the solution of Problem 1 exists, then it is unique in $L_p(\Omega)$, $1 < p \leqslant 2$ if $f(r) \in T(\Omega)$.

No analogous uniqueness theorem has successfully been proved here in the case of complex $\varkappa$.

3. Representing the right side of (1.1) by the Kantorovich-Lebedev integral, let us limit ourselves to the case $f_n(r) = I_\eta(\varkappa r) I_\eta^{-1}(\varkappa \alpha_{2n})$ and let us seek the solution (1.1) as a series ($x_k(n)$, $y_k(n)$ are constants to be determined)

$$\frac{\rho}{A} q_n(\rho) = x_0 \frac{I_\eta(\varkappa\rho)}{I_\eta(\varkappa\alpha_{2n})} + \sum_{k=1}^{\infty} \left[x_k(n) \frac{I_{-iz_k}(\varkappa\rho)}{I_{-iz_k}(\varkappa\alpha_{2n})} + y_k(n) \frac{K_{-iz_k}(\varkappa\rho)}{K_{-iz_k}(\varkappa\alpha_{2n-1})} \right] \quad (3.1)$$

$$\alpha_{2n-1} \leqslant \rho \leqslant \alpha_{2n}, \qquad n = 1, 2, \ldots, N$$

We insert (3.1) into the left side of (1.1) and representing the kernel $k(r, \rho)$ in the form in [1], we integrate in the series. We hence arrive at an infinite system of linear algebraic equations to determine the constants, one of which is presented below (another can easily be obtained and has an analogous form)

$$A_{11}(m) X(m) + A_{12}(m) Y(m) + \sum_{n=m+1}^{N} [G_{11}(n) X(n) + G_{12}(n) Y(n)] = B_1(m) \quad (3.2)$$

$$A_{12}(m) = \{a_{rl}(1, 2)\} = \frac{iW[K_{-iz_l}(\lambda_{2m}), K_{-i\zeta_r}(\lambda_{2m})]}{(\zeta_r^2 - z_l^2) K_{-iz_l}(\lambda_{2m}) K_{-i\zeta_r}(\lambda_{2m})}$$

$$A_{11}(m) = \{a_{rl}(1, 1)\} = \frac{iW[I_{-iz_l}(\lambda_{2m}), K_{-i\zeta_r}(\lambda_{2m})]}{(\zeta_r^2 - z_l^2) I_{-iz_l}(\lambda_{2m}) K_{-i\zeta_r}(\lambda_{2m})}$$

The notation from [1] is used here. Because of the awkwardness of the expressions obtained, the matrix elements $G_{11}(n)$, $G_{12}(n)$ are not presented.

The equivalence between the infinite system (3.2) and the integral equation (1.1) and the converse is established in appropriate classes of functions and sequences because of the property of minimality of the system of functions $\{I_{\lambda_k}(z), K_{\lambda_k}(z)\}$ [1].

In those cases when uniqueness holds for the integral equation, uniqueness also follows at once for the infinite system. If it turns out that the infinite system is equivalent to an equation with a completely continuous operator, then solvability of the system follows immediately.

It is easy to verify that the elements of the matrix $A_{kk}(m)$ $(m = 1, 2, \ldots, N)$ tend to elements of the matrix $A = \{(\zeta_r - z_l)^{-1}\}$ as $|\zeta_r| \to \infty$, $|z_l| \to \infty$, while the elements of the matrices $A_{kj}(m)$, $G_{kj}(n)$ decrease exponentially.

4. Using the inverse matrix A^{-1} [7], the system (3.2) can be written in normal form, where the first matrix equation is

$$X(m) = A^{-1}B_1(m) + A^{-1}[A - A_{11}(m)]X(m) - A^{-1}A_{12}(m)Y(m) - \sum_{n=m+1}^{N} A^{-1}[G_{11}(n)X(n) + G_{12}(n)Y(n)] \tag{4.1}$$

Using the estimate (1.3), it can be established that the matrix elements of the right side of the system (4.1) generate completely continuous operators in the space of the sequences $s(\sigma)$, $0 < \sigma \leqslant 1/2$ with the norm

$$\|X\|_{s(\sigma)} = \sup_l |l^\sigma x_l|, \quad \lim |l^\sigma x_l| = 0 \quad (l \to \infty)$$

Because of the above, the single-valued solvability of the system for $\varkappa > 0$ in $s(\sigma)$ results from the single-valued solvability of (1.1) because of the positive definiteness of the operator of the left side in $H_{0.5}(\Omega)$ and the minimality of the system of functions $I_{\lambda_k}(z)$, $K_{\lambda_k}(z)$.

The following theorem is valid.

Theorem 4. The solution of (1.1) taken in the form (3.1) belongs to the space $C(0.5)$.

The proof of the theorem follows from the validity of the imbedding $C(0.5) \subset L_p(\Omega)$ $1 < p < 2)$ and that the solution taken in the form (3.1) belongs to the $L_p(\Omega)$ mentioned, if $X \in s(\sigma)$, $Y \in s(\sigma)$.

It follows from the theorem that the solution of (1.1) can be sought in the form (3.1) and this solution will be unique in the case $\varkappa > 0$ because of Theorem 3. For complex values of $\varkappa$ the single-valued solvability of (1.1) has not been proved successfully by the methods mentioned. However, in the general case the system (4.1) turns out to be quasi-regular and known methods can be used to investigate it [8].

In the case of sufficiently small wedge angles, the operators of the right side of (4.1) become compressive and the system can be solved by successive approximations.

5. Assuming the wedge angle to be small, let us investigate the system (4.1) in a zero approximation by selecting $X^{(0)}(m) = A^{-1}B_1^{\cdot}(m)$, $Y^{(0)}(m) = A^{-1}B_2(m)$ as the last matrix. Evaluating the elements of the matrix $X^{(0)}(m)$, $Y^{(0)}(m)$, we find

$$x_l^{(0)}(m) = x_m(z_l) - \frac{1}{2\eta K_+'(-z_l)}\left[\frac{S^-(\eta, m)}{(z_l + i\eta)K_-(i\eta)} - \frac{S^+(\eta, m)}{(z_l - i\eta)K_+(i\eta)}\right] + O\left(\frac{\pi}{\alpha}d^{-\pi/\alpha}\right)$$

$$S^+(\eta, m) = V(\eta, m) + \eta\left[1 - \sum_{k=m+1}^{N} U(\eta, k)\, d_{km}^{1-\eta}\right] \tag{5.1}$$

$$S^-(\eta, m) = V(\eta, m) - \eta, \quad V(\eta, m) = I_\eta'(\lambda_{2m})\, I_\eta^{-1}(\lambda_{2m})$$

$$U(\eta, k) = I_\eta(\lambda_{2k-1})\, I_\eta^{-1}(\lambda_{2k})$$

$$d_{km} = \alpha_{2k-1}/\alpha_{2m}, \quad d = \inf_n \{\alpha_{2n}/\alpha_{2n-1}\}$$

The expressions for $y_l^{(0)}(m)$ are analogous in form. Let us note that the expressions for $x_m(z_l), y_m(z_l)$ can be written as

$$x_m(z_l) = x(z_l) + \Delta_l(m), \quad y_m(z_l) = y(z_l) + \delta_l(m)$$

Here $x(z_l), y(z_l)$ is a solution of the system (4.1) in the case of just one stamp acting in the domain $\alpha_{2m-1} \leqslant r \leqslant \alpha_{2m}$ [1], and $\Delta_l(m), \delta_l(m)$ are corrections to the solution of the problem with one contact domain which characterizes the influence of the remaining $N-1$ stamps of the contact domain mentioned.

Inserting (5,1) into (3.1) and summing the series obtained in the contour integral, we obtain the following asymptotic formulas to solve (1.1):

$$\frac{\rho}{A} q_m(\rho) = x_0 \frac{I_\eta(\varkappa\rho)}{I_\eta(\lambda_{2m})} - \frac{1}{2\pi i} \int_\Gamma \left[x_m(-t) \frac{I_{it}(\varkappa\rho)}{I_{it}(\lambda_{2m})} + \right. \tag{5.2}$$

$$\left. y_m(-t) \frac{K_{it}(\varkappa\rho)}{K_{it}(\lambda_{2m-1})} \right] \frac{K_+'(t)}{K_+(t)} dt$$

$$m = 1, 2, \ldots, N, \qquad \alpha_{2m-1} \leqslant \rho \leqslant \alpha_{2m}$$

Here the contour Γ is in the lower half-plane enveloping the zeros and poles of the function $K_+(t)$ from above and the points $t = \pm i\eta$ from below.

The expression (5.2) is easily calculated by using operational calculus formulas when asymptotic estimates of the behavior of the Bessel functions $I_\lambda(\varkappa\rho), K_\lambda(\varkappa\rho)$ are used for large values of $|\lambda|$ [9]. Then the asymptotic expression for the contact stresses in the case of Problem 2 (for $\rho \to \alpha_{2m-1}$, say) will have the following form

$$q_m(r) \sim D_1(\eta, m) [1 - (\rho / \alpha_{2m-1})^{-\pi/\alpha}]^{-1/2} [1 + O(\ln \rho / \alpha_{2m-1})]$$

$$(\rho \to \alpha_{2m-1})$$

$$D_1(\eta, m) = \left(\frac{A\alpha_{2m-1}}{2\eta \sqrt{\pi C}} \right)^{-1} \left[\frac{S^-(\eta, m)}{K_-(i\eta)} - \frac{S^+(\eta, m)}{K_+(i\eta)} \right]$$

It is seen that the function $q(\rho)$ belongs to the space $C(0.5)$ for $\rho \in \Omega$.

In the case of Problem 1, the solution of (1.1) depends on two arbitrary constants $C_1(m), C_2(m)$ found from the conditions of boundedness of the displacements at the points

$$\rho = \alpha_{2m-1}, \quad \rho = \alpha_{2m}, \quad m \quad 1, 2, \ldots, N$$

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ON THE PROBLEM OF THE ELASTIC STABILITY OF A LOCALLY LOADED CYLINDRICAL SHELL (SUPPLEMENT)

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An approximate analytic representation of the solution of a nonlinear equation describing the subcritical axisymmetric shell bending was used in [1] in an investigation of the stability of equilibrium of a semi-infinite circular cylindrical shell loaded by a uniform radial stress resultant along a hinge supported edge. In substance, this representation corresponds to the two first terms of the expansion of the desired nonlinear solution in a power series of the parameter

$$p = \mu q = 2Q \,/\, (Eh\mu^2) \left(\mu^2 = h \,/\, [R \sqrt{3(1-\nu^2)}]\right)$$

Here Q is the intensity of the external radial stress resultant, h and R are the shell thickness and radius, and E and ν are the Young's modulus and Poisson's ratio of the shell material. The construction of higher approximations was nor carried out because of their extreme awkwardness.

However, the desire to solve more exactly the stability problem formulated in [1] forced the authors to return to the question of refining the solution of the nonlinear boundary value problem of subcritical shell bending. To solve this problem, the procedure of differentiating with respect to the parameter was used in combination with the method of finite differences.

Differentiating (1.1) from [1] with respect to the parameter p and later going over to finite differences yields the following successive approximations process to the desired nonlinear solution (the meaning of the notation is disclosed in [1]): if the functions $\eta_i(x)$ and $\vartheta_i(x)$ are a solution of the nonlinear problem for $p = p_i$, then the functions

$$\eta_{i+1}(x) = \eta_i(x) + \Delta p \eta_i^{\cdot}(x), \quad \vartheta_{i+1}(x) = \vartheta_i(x) + \Delta p \vartheta_i^{\cdot}(x)$$

are the approximate solution for $p = p_{i+1} = p_i + \Delta p$, where $\eta_i^{\cdot}(x)$ and $\vartheta_i^{\cdot}(x)$ are determined as a result of solving the linear boundary value problem